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# Prior Information for Nonlinear Modelling of Tokamaks

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# Contents

**Abstract**

This report is concerned with the construction of abstract mathematical models for experimentally observed tokamak behaviour, and their identification with experiment. Simple nonlinear models constrained by symmetry are proposed for different patterns of tokamak behaviour, viz. (1) tearing modes, (2) regular sawteeth oscillations , (3) fishbones, (4) edge-localised modes (ELMs) and control, (5) resistive wall modes (RWMs) and (6) compound sawteeth. Other physics-based models are also catalogued. Techniques for comparing and in some cases fitting the models to tokamak time series data are also briefly discussed, and the implications of successful comparison work highlighted.

# Chapter 1

# Introduction

## 1.1 From Linear to Nonlinear Modelling

Most experimenters are familiar with the abstract mathematical analysis of of exponential growth in linear systems. Besides direct growth, it is also possible to have overstability, meaning that a signal oscillates as it grows. The two different mathematical models, of direct and overstable growth, are distinguished in two ways. First, at a qualitative level, overstability is distinguished by the fact that the growing signal requires two variables to specify its behaviour as a function of time, namely its amplitude and phase of oscillation, whereas for direct growth only amplitude is needed. This is the way whereby the two cases are separated by their topology, as given by their dimensionality. The second and more quantitative way, is to compute a growth rate and frequency for the signals, when zero frequency plainly indicates the model with direct growth.

One of the aims of this report is to indicate how these two ways, topological and quantitative, of modelling linear systems can be extended to nonlinear ones. Nonlinear systems near onset may be much more complicated because (1) there are different possible nonlinearities, and (2) the nonlinearity may couple more than one almost-unstable modes together. To assist the modelling, it is helpful to gather as much prior information regarding the nonlinearities, which is the main aim of this report.

In fact, even the linear response may be more complicated that the simple direct/overstable picture above [1, § 6.3]. In the practical analysis of experiment where instability may occur at a time late in the shot, signal growth is affected by the way in which the threshold of instability is crossed, as will be discussed further below. This effect needs disentangling from the nonlinear ones. However, an example of the benefits of nonlinear analysis is that information can be gained about the subcritical response of the system, as the following explains. Although the growth rate is inevitably zero as the stability threshold is crossed, the coefficient of the leading order nonlinear term does not change sign, and indeed will not normally change by much as the system goes from subcritical to critical. Other benefits are discussed below.

## 1.2 Background and History

The nonlinear analysis of tokamak behaviour is not an easy subject [1], for the following reasons:

1. Tokamaks are inherently nonlinear devices because of the need to balance the pressure of the plasma by the magnetic force, which is quadratic in the field
2. Tokamaks are presently pulsed devices, so it is questionable whether discharges are ever truly in a steady state. Moreover additional heating and fuelling processes may generate large and sometimes long-lived transients.
3. Accurate measurement of plasma properties tends to be difficult even at temperatures of a few $eV$ (electron Volts) and is certainly not any easier in fusion relevant conditions.
4. There is small-scale turbulence in tokamaks which is significant not least because it apparently determines energy confinement properties.
5. Tokamaks are subject to active feedback control to prevent unwanted vertical motion and to control the shape of the plasma.

The formidable difficulty of tokamak modelling has however been matched by formidable developments in the theory of nonlinear systems in the past $20-25$ years. The nonlinear behaviour of deterministic systems close to instability has been intensively studied and catalogued in textbooks such as Kuznetsov [2]. Methods for fitting nonlinear dynamical models to experimental time series have been devised and used successfully to fit data, see the handbook [3]. The effects of “noise”, namely high frequency components of signals such as might be generated by small-scale turbulence, have been studied, so that appropriate averaging and processing techniques now exist, see for example the textbook by Tong [4]. The control theory of nonlinear dynamical systems has also been extensively developed as demonstrated by the textbook [5]. Hence, the prospects for successful, nonlinear modelling of tokamaks now look extremely good. There are moreover, simplifying factors as far as tokamak modelling is concerned, namely that tokamak equilibria are rotationally and reflectionally symmetric [6] and, over a usefully large parameter range exhibit no large-scale instability of any kind. In addition, in respect of reason 2 above, for many instabilities, the growth timescale is such that the base discharge is effectively in steady-state.

The latter property is clearly desirable for economic power production, however in order to understand tokamak behaviour better with the aim of ultimately improving device performance, it is desirable to be able to study the response of the device to perturbations. In any event, optimal device performance is often found close to stability boundaries. Most tokamak control systems [7, 8] work on a timescale set by the electrical properties of the magnetic coils which is of the order of milliseconds, whereas the plasma perturbations may grow in times of less than a microsecond. Hence the behaviour of perturbations to the equilibrium can have significant implications for reliable operation and needs to be understood.

The fact that interest attaches to the regime where the tokamak is close to instability should be helpful, in that it might be expected that nonlinearity is relatively weak. It must be admitted that even this is not clear, for models of the large-scale effect represented by the tearing mode instability, involve the singular effect of current sheet formation, and the role of the small-scale turbulence may not be entirely passive in the development of this and other instabilities. However, it is necessary to make choices, and that made here concerning the weakness of the nonlinearity is conservative in that such models have found successful application in laser physics [9] and hydrodynamic stability [10], to name but two areas. See also ref [11] for an application of nonlinear dynamical systems techniques to an industrial control problem, and more abstract references such as refs [12, 13].

## 1.3 Contents

The study of linear systems with a stochastic element (to account for turbulence and other features) is an extensive subject in its own right. Some important results and their implications are given in Section 2. The principal part and novelty of the present paper, Section 3 is concerned with what Tong [4] refers to as the 'skeleton' of a stochastic model, namely the 'bare bones', deterministic component. Here, models consisting of a small number of coupled ordinary differential equations (ODEs) are proposed for a wide range of phenomena exhibited by the tokamak. (Many of which will also apply to the reversed field pinch fusion device, which shares the same symmetry group properties.)

Of course, theories and models must be testable against experiment and so it is necessary at minimum, to describe how this can be done. Section 4 outlines some of the practical details involved in the application of the analysis to tokamak time series, especially the use of delay coordinates. It includes discussion of both topological methods (Section 4.1) for characterising time series before (Section 4.2) describes methods for quantifying tokamak behaviour in terms of the weakly nonlinear models introduced in Section 3. Lastly Section 5 provides a summary.

# Chapter 2

# Summary of Linear Results

Linear stochastic modelling is dominated [4] by the auto-regressive/moving average (ARMA) models, sometimes referred to as Box-Jenkins after the standard textbook on the subject [14]. ARMA models and their variants (eg. ARIMA) do not in general represent very well systems with obvious nonlinearities which includes time series exhibiting 'jumps'. Moreover, the usual, Gaussian ARMA models cannot reproduce time series unless the data are reversible in a statistical sense, ie. look the same when plotted against $(-t)$ rather than positive time $t$. However such models, which are available as part of most statistical packages, might usefully be employed to characterise the 'flat top', to quantify 'noise' or randomness and perhaps to determine to what extent any fluctuations are intrinsic to the tokamak rather than say to thermal noise in certain types of detector.

The behaviour of linear deterministic models, and to a large extent stochastic models, may be understood by considering the evolution equation

$$\frac{da}{dt} = \dot{a} = \gamma(t)a + \zeta(t) \tag{2.1}$$

where $a$ is a measured quantity for which the time dependent behaviour is of interest, $\zeta(t)$ is typically a small forcing term, and growth rate $\gamma(t)$ is function of time which begins negative but may subsequently become positive, eg.

$$\gamma = \gamma_0.(t - t_0) \tag{2.2}$$

for positive constants $\gamma_0$ and $t_0$. In fact, when $\gamma < 0$ and $\zeta$ is a random forcing function with Gaussian statistics, Equation (2.1) represents a very simple ARMA model. Regarding $\zeta(t)$ as deterministic, Equation (2.1) has the well-known solution

$$a(t) = M(t)^{-1}\left(a_0 + \int_0^t M(t')\zeta(t')dt'\right) \tag{2.3}$$

where the integrating factor

$$M(t) = \exp\,(-\int_0^t \gamma(t')dt') \tag{2.4}$$

The most important feature with implications for model building is probably the time integral in Equation (2.4), which indicates that this system has a 'memory'. If there is a significant time over which $\gamma$ is negative, say $t_0 > 1$ in the example Equation (2.2), then $\gamma$ must be positive for a comparable length of time before its time integral becomes positive so that $a(t)$ starts to grow. This is also referred to as the 'slow passage' effect. A more intuitive aspect of system behaviour is that if the growth-rate is negative, the initial condition quickly becomes unimportant and the long term behaviour is determined by $\zeta$ (a feature which is in fact also true of ARMA models).

As far as topological analysis is concerned, the effect of 'noise' in its various forms (initial, additive and forcing) is unimportant, almost by definition. Even quantitatively, there seem to be few significant departures from what the deterministic models predict [15, § 8], even when the skeleton equation is nonlinear. The only major impacts noted are that 'initial' noise may reduce or eliminate the slow passage effect, and that it may significantly affect the behaviour of periodic solutions which have two disparate time-scales of behaviour (hetero-/homo-clinic orbits) [16].

An important feature may be demonstrated by considering the second order version of Equation (2.1) in which all quantities are replaced by 2-vectors, eg. $a \rightarrow \mathbf{a} = (a, b)$, except that $\gamma$ becomes a $2 \times 2$ matrix. Now, if $\gamma$ is a normal $2 \times 2$ matrix, then it may be diagonalised, and if both its eigenvalues are negative, each eigenvector will separately decay. (Note that matrices and operators are herein referred to as normal when most authors would use the term symmetric - in this paper 'symmetric' is reserved for describing spatio-temporal symmetries.) This should be familiar behaviour, but it is worth underlining the fact that this implies that the amplitude of the solution $|\mathbf{a}(t)|$ will at all subsequent times be less than its initial value at $t = 0$.

Consider instead what happens if $\gamma$ takes the form

$$\gamma = \begin{pmatrix} -\gamma_1 & 1 \\ 0 & -\gamma_2 \end{pmatrix} \tag{2.5}$$

then straightforward manipulation shows that

$$\frac{d(a^2 + b^2)}{dt} = ab - \gamma_1 a^2 - \gamma_2 b^2 \tag{2.6}$$

implying that $|\mathbf{a}(t)|$ grows initially for most positive initial values of $a(0)b(0)$ provided $\gamma_1$ and $\gamma_2$ are small compared to 1. In fact, the analytic solution of the second order system shows that $a$ may increase by a factor of order $1/(\gamma_1 - \gamma_2)$ before decay sets in. (Equation (2.5) is of course maximally non-normal (non-symmetric) when $\gamma_i = 0$, but it is worth remarking that extreme non-normality does not guarantee growing $|\mathbf{a}(t)|$, eg. change 1 to $(-1)$ in the top right matrix entry.) It may be shown that the transient growth phenomenon persists regardless of the order of the matrix [17, § 14], and so applies to matrices which better approximate spatio-temporal differential operators (ie. PDE models). It might be objected however that the time taken to reach maximum is long, also of order $1/\gamma_i$, and other, stochastic effects might intervene.

Tong [4, § 2.8] also describes linear models containing delays, eg.

$$\dot{a}(t) = \gamma . a(t - \Delta t) \tag{2.7}$$

where $\Delta t > 0$. Such models could represent aspects of tokamak behaviour such as the time for the heat pulse associated with the sawtooth crash to propagate across the minor radius to drive another mode unstable, or the response time of a control system. Fortunately, by use of models based on delay co-ordinates as recommended in Section 4, such effects can in principle be captured. As might be inferred from the fact that delay co-ordinates can be used to represent ODEs, solutions to delay-differential equations, both linear and nonlinear, may be represented in terms of solutions to ODE models as explained in ref [18, § 7].

For application to plasma dynamics below it is helpful to consider the second order equation

$$\frac{d^2a}{dt^2} = \gamma(t)a \tag{2.8}$$

To exhibit analytic solution to Equation (2.8), assume a linear dependence of $\gamma$ upon $t$ as in Equation (2.2) and choose the origin of time such that $t_0 = 0$. It is convenient to take $\gamma_0 = \epsilon^3$, then, in terms of the normalised time $\tau = \epsilon t$, the amplitude $a$ satisfies:

$$\frac{d^2a}{d\tau^2} = \tau a \tag{2.9}$$

ie. $a$ satisfies Airy's equation. Airy's equation has two solutions, one of which $Ai(\tau)$ damps exponentially, and the second of which $Bi(\tau)$ may be expressed as a sum of Bessel functions [19, § 10.4]:

$$Bi(\tau) = \sqrt{(\tau/3)}[I_{-\frac{1}{3}}(\frac{2}{3}\tau^{\frac{3}{2}}) + I_{\frac{1}{3}}(\frac{2}{3}\tau^{\frac{3}{2}})] \tag{2.10}$$

For large $\tau$, $Bi$ increases as $\exp(\frac{2}{3}\tau^{\frac{3}{2}})$,

# Chapter 3

# Prior Information for Nonlinear Systems

## 3.1 General Considerations

The results of the previous section Section 2 and the symmetry arguments of ref [6] suggest the following guidelines in the construction of candidate nonlinear 'skeletons' for modelling tokamak behaviour, starting with a candidate instability mechanism. The key questions to ask of the instability, which almost invariably will be described by a linear theory, are

1. Is the instability direct (ie. real growth rate) or overstable (ie. pure imaginary growth at critical)?
2. Is the linearised operator governing stability normal (symmetric)?
3. What spatio-temporal symmetry properties are likely to govern the nonlinear evolution of the instability?

Consistent with earlier statements about the difficulty of modelling tokamaks, even Question 1 is not necessarily easy to answer. For example, in the case of tearing modes, simple MHD theory implies perturbations should have direct growth, but a more complicated theory taking into account 2-fluid effects also predicts diamagnetic rotation, ie. overstability, see ref [1], and including pressure effects in MHD also leads conditionally to overstability.

Question 2 can be answered affirmatively when the instability is ideal MHD. It also may be possible to decide that the linear operator governing an instability is non-normal on the basis of the observed behaviour of time series. Lack of operator normality implies subcritical perturbations may grow to large values before ultimately decaying (Section 2), hence they may make it hard to determine exactly when the stability boundary has been crossed, ie. make for a soft transition to instability. However a normal stability operator will make for a hard transition because subcritical perturbations are forced to decay, but when the instability threshold is crossed there may be an apparent pause before explosive growth eg. as $\exp(\gamma_0 t^2/2)$ occurs, thanks to the 'memory effect' which applies to nonlinear systems as well as linear [20].

Question 3 has been considered in part in ref [6]. It is expected that rotational symmetry will be important for all primary instabilities of a tokamak equilibrium, and a further reflectional symmetry will be significant if the model is MHD. Certain particle or kinetic effects may also have additional symmetry if say, they lead to a rigid rotation of the plasma (cf. ref [21]) or occur in a rigid reference frame where the electric field vanishes, hence the symmetry will also govern their subsequent nonlinear development.

The answers to the questions are not necessarily independent. Results from other fields of physics may be enlightening. Stability analysis of sheared mean flow in the Navier-Stokes equations usually leads to non-normal stability operators [22, § 1.4], suggesting for example that instabilities involving neutral beam driven rotation will be 'soft'.

## 3.2 Nonlinear Results

Much of the mathematical theory of nonlinear systems is qualitative, rather than quantitative, because of the difficulty of finding analytic solutions to even the simplest of nonlinear equations, eg. ref [23, § XIV]. Moreover, so accustomed are most people to dealing with linear systems, it is worth re-iterating that the amplitudes of nonlinear solutions are not generally arbitrary in the way that linear solutions are, and of course it is not normally possible to superpose two nonlinear solutions to construct a third nonlinear solution.

The qualitative (precisely topological) mathematical theory of the onset of instability in nonlinear systems is known as bifurcation theory, see for example Kuznetsov [2]. The key idea is that near onset, system behaviour is governed by a small number of coupled ordinary differential equations (ODEs), with nonlinear interactions among the variables represented by low order terms in a multi-variable Taylor expansion, ie. polynomials. It is a natural generalisation of linear stability theory which can be regarded as a truncation of the Taylor series at first order. Frequently the time dependent variables represent mode amplitudes.

Equivariant bifurcation theory (EBiT) means bifurcation theory analyses performed in the presence of symmetry [24, 10]. An example frequently quoted concerning the effect of symmetry is when the problem is invariant under reflection. For then both $(+a)$ and $(-a)$ must be solutions of the ODEs, which rules out terms such as $a^2$ in the polynomials, which do not change sign when $a$ does. The general rule is that the effect of symmetry is robust, in that 'noise' in its various forms does not change the qualitative behaviour of models, except as noted above. Reference [25] has examined the value of using equivariance when modelling dynamical systems.

Despite the above caveats, it is worth recording two common cases where useful analytic results are available. The first case is an equation of the general (Bernouilli) form

$$\dot{a} = \gamma(t)a + s(t)a^n \tag{3.1}$$

where $n \neq 1$, which may be linearised by the transformation $z = a^{1-n}$. The resulting equation is

$$\dot{z} = -\tilde{\gamma}(t)z - \tilde{s}(t) \tag{3.2}$$

where

$$\tilde{\gamma} = (n-1)\gamma(t), \qquad \tilde{s} = (n-1)s(t) \tag{3.3}$$

Equation (3.2) may be solved by means of the integrating factor as in Section 2. Applying this procedure to Equation (3.1) with $n = 2$, setting $\gamma = \gamma_0 t$, writing $\gamma_0 = \epsilon^2$, and introducing normalised time $\tau = \epsilon t$, the solution is

$$a = a_0 \exp{(\tau^2/2)} / \left(1 - \frac{a_0}{\epsilon} \int_{t'=0}^{t'=\tau} s(t') \exp{(t'^2/2)} dt'\right) \tag{3.4}$$

where $a(0) = a_0$.

Elliptic functions are solutions of the equation

$$\dot{a}^2 = 1 + \gamma a + \mu_1 a^2 + \mu_2 a^3 + \mu_3 a^4 \tag{3.5}$$

ie. where the squared derivative of $a$ is proportional to a quartic in $a$. By suitable transformation, the quartic may be put in the form

$$\dot{b}^2 = (1 - b^2)(1 - mb^2) \tag{3.6}$$

which is defined to have the Jacobi elliptic functions as solutions, for example the analogue of sine

$$b = sn(t + t_0|m) \qquad m < 1 \tag{3.7}$$

$$= m^{-1/2} sn(m^{1/2}[t + t_0]|1/m) \qquad m > 1 \tag{3.8}$$

Each of the trigonometric functions has a Jacobi elliptic function analogue. For moderate values of $m$, the elliptic functions $sn$ and $cn$ resemble sin and $\cos$ closely. However, the period of the elliptic function depends on $m$ and becomes infinite as $m \to 1$, while the graphs acquire steeper gradients, so that $cn$ becomes very spiky and $sn$ flips between $(-1)$ and $(+1)$ [26, § 63].

## 3.3 Suggested Models

The standard EBiT approach to representing bifurcations in a rotationally symmetric system is as follows [6]. Introduce an explicit spatial dependence, by supposing that the angle about the axis of symmetry is $\phi$, then symmetry-breaking solutions $y(t)$ may be written

$$y = a \exp{(iK\phi)} + \bar{a} \exp{(-iK\phi)} \tag{3.9}$$

Here the overbar denotes complex conjugate, $K$ is (integer) mode number and $a(t)$ is the time dependent complex mode amplitude. Now, translating the angle $\phi$ by $p/K$ in Equation (3.9) shows that if $a$ gives a solution to the problem, $ae^{ip}$ must also be a solution. This means that model equations such as $\dot{a} = a^2$ are not possible, instead it is necessary that $\dot{a} = f_1(|a|^2)a$ [27, § 12].

To prepare the models below for comparison with experiment, it will generally be necessary to treat the linear growth rate $\gamma$ as a time dependent variable in its own right, either by increasing the order of the ODE as in Section 3.3.3 or specifying $\gamma(t)$ explicitly as in Section 3.3.4. The models must also be expressed in terms of real, ie. not complex, variables and parameters.

### 3.3.1 Tearing modes

The case of tearing modes is interesting because there is a well-developed theory of how they should develop in time as a function of discharge parameters. Physical interest attaches to the island width $w = \sqrt{|a|}$ where $a$ is mode amplitude and a multiplicity of papers have derived an evolution equation for $w$. To contrast island width evolution models with the EBiT models, it helps to describe the latter first. Assuming resistivity is important, EBiT models take the form of a Landau equation [28, § 8.3]

$$\dot{a} = f_1(|a|^2)a = \gamma a + \mu_1|a|^2a + \mu_2|a|^4a \tag{3.10}$$

where $f_1$ is an arbitrary, smooth complex function of a real positive variable, expressed as a truncated Taylor series in Equation (3.10). Writing $a = r\exp(i\xi)$, where $r$ is the (real) amplitude of complex number $a$ and $\xi$ is its phase, Equation (3.10) gives

$$\dot{r} = \gamma_r r + \mu_{1r}r^3 + \mu_{2r}r^5 \tag{3.11}$$

$$\dot{\xi} = \gamma_i + \mu_{1i}r^2 + \mu_{2i}r^4 \tag{3.12}$$

If a reflectionally symmetric model such as MHD is appropriate, then $\bar{a}$ must also be a solution of Equation (3.10) and so the parameters $\gamma$, $\mu_1$ and $\mu_2$ must be real, implying that there is no rotation, $\dot{\xi} = 0$. This simpler model may also be appropriate to the case where the plasma rotates rigidly, ie. there is a frame rotating with angular velocity $\dot{\xi} = -\mu_i$ in which the mode is stationary. Note there is no constraint on the signs of the parameters, although unless one has a sign different from the other two, the dynamics are either complete decay or rapid blow-up.

As far as topological analysis is concerned, $\mu_2$ is irrelevant unless $\gamma$ and $\mu_1$ vanish. This point is important because it highlights the fact that different models might have to be used to fit system behaviour qualitatively (ie. topologically) from quantitatively. Although the topological model is relatively inconsequential for the case of a single mode, this will not be true for the multiple mode interactions discussed below.

In quantitative terms, the above model has theoretical support only in the limit where the magnetic field is weak (small Lundquist number), which is emphatically not the case in tokamaks, where singular, current sheet type behaviour predominates. The mode radial dependence may most accurately be expressed [29] in similarity form as $f_{r^*}(r^*/w(t), r^*, t)$, where $r^*$ is the device radial coordinate. The form of $f_{r^*}$ where the spatial structure changes at least as rapidly as the mode amplitude (since $w = \sqrt{|a|}$)

implies there is no appreciable phase of linear growth. The model equations devised from first principles theory take the form [30, § 5.4]

$$\dot{w} = \mu_{00} + \mu_{01} w + \mu_{02} w^{n_2} \tag{3.13}$$

where $n_2$ is an integer power. It is normally assumed that such modes rotate rigidly. Note that for the choice $n_2 = -1$, by transforming $r^2 \to a$, Equation (3.13) can be re-expressed as $\dot{a}$ is equal to a polynomial in $\sqrt{a}$, which may be more convenient for quantitative fitting purposes.

Finn and Sovinec [31, Appendix] have put forward a second order model for the non-linear behaviour of tearing modes. The evolution of a non-ideal $m = n = 1$ mode does not involve singular behaviour and Firpo and Coppi [32] have derived from first principles a model, their Equation (10), of the form Equation (3.10) in which $f_1$ is a rational polynomial.

Experimental efforts [33] have been made to distinguish whether singular behaviour is significant in practice. Ref [34] describes an exponentially growing tearing mode.

### 3.3.2 Regular sawteeth oscillations

Assuming resistivity is unimportant, EBiT models for MHD are derivable from the Lagrangian

$$2L(a, \dot{a}, t) = |\dot{a}|^2 + f_2(|a|^2) = |\dot{a}|^2 + \gamma|a|^2 + \mu_1|a|^4 + \mu_2|a|^6 \tag{3.14}$$

where $f_2$ is an arbitrary, smooth real function of a real positive variable, expressed as a truncated Taylor series in Equation (3.14), and $\gamma$ and the $\mu_i$ are real, possibly time-dependent parameters. The real ODEs corresponding to Equation (3.14) are found, by setting $a = r \exp{(i\xi)}$ and using the variational principle [6], to be

$$\ddot{r} - r\dot{\xi}^2 = \gamma r + 2\mu_1 r^3 + 3\mu_2 r^5 \tag{3.15}$$

$$\dot{\xi} = C/r^2 \tag{3.16}$$

(Note that the solutions of Eqs (3.15)(3.16) may be expressed in terms of elliptic functions, where in some cases $r^2$ is an elliptic function rather than $r$.) The model for the sawtooth is completed by an equation for the amplitude $b$ of the axisymmetric state.

$$\dot{b} = \nu_1 - \nu_2 b^2 - r^2 \tag{3.17}$$

As far as topological analysis is concerned, $\mu_2$ is again irrelevant, unless $\gamma = 0$ and/or $\mu_1 = 0$. If small amounts of resistivity are present, the relevant, topologically valid model is expected to be of Takens-Bogdanov (TaB) type, see below. Reference [35] discusses the theoretical issues involved.

Assuming the Casimir $C = 0$, the paper by Callen *et al* [36] provides experimental evidence that $r \propto \exp{(t^{3/2})}$, consistent with the linearised version of the model Equation (3.15) if $\gamma \propto t$ as expected, eg. [1, § 6.3].

A third-order ODE model of the sawtooth has been put forward by Thyagaraja *et al* [37]. Earlier, Haas and Thyagaraja [38] constructed a simpler model of the sawtooth which they described as being of predator-prey form.

### 3.3.3 Edge-localised modes (ELMs)

If these are ideal, non-singular MHD modes, then the EBiT model has to be identical to that for amplitude $r$ as in Equation (3.15). The difference is that the coupling of $b$ back on $r$ might be more important, ie. $\gamma = \mu_0(b) = \mu_{00} + \mu_{01}b + \mu_{02}b^2$. Again, if ELMs are weakly resistive modes or not well described by MHD, then the more general TaB models need to be considered, see below.

Horton and Sugama [39] have suggested that their third order model for the L-H transition also models ELMs. They later derived a sixth order model [40] by modal truncation, as well as a third order, energy balance model. Pogutse et al [41] also present a modally truncated model and a semi-phenomenological energy-balance model of low order. Lebedev et al [42] present a third order model and discuss a possible relationship with the van der Pol oscillator. Earlier, Lebouef et al [43] presented a predator-prey model for ELMs.

### 3.3.4 Fishbones

'Fishbones' describes signals which exhibit bursts of high frequency behaviour, separated by intervals of relative calm. Zonca *et al*[44] have proposed a predator-prey system as a model for fishbones, which might be preferred for quantitative fitting purposes because of its theoretical underpinnings.

The topological characteristics of intermittent behaviour such as fishbones may be classified according the scheme proposed by Golubitsky *et al* [45]. The basic idea is that the growth rate parameter $\gamma$ varies on as slow timescale, as

$$\gamma(t) = A + B\cos(\epsilon t) \tag{3.18}$$

where $A$ and $B$ are constants and $\epsilon$ is a small frequency. The most likely model in Golubitsky *et al*'s scheme for fishbones is the co-dimension one burster, where $A = 0$ since this occurs both in the systems Equation (3.11) and for TaB. The EBiT approach would see the cosine replaced by coupling to a nonlinear oscillator to represent the tokamak sawtooth.

The review by Knobloch *et al* [46] is a less abstract, more physically based classification of models for bursting behaviour. However, the $D_4$-symmetric model which is prominent in ref [46] is likely to apply only when hard, confining boundaries are important for mode structure.

### 3.3.5 Non-axisymmetric Phenomena

In order to control ELMs, coils have been to added to some tokamaks which break the rotational symmetry. It is unclear how far their influence will extend into the plasma. For most phenomena, except ELMs, the conclusions of Crawford and Knobloch [47] are comforting, in that the effects of small symmetry-breaking perturbations of the governing ODEs become less noticeable as the mode amplitudes increase.

In the case of ELMs subject to control, the only persisting invariance might be the reflectional $Z(2)$ symmetry, depending on how the coils are wound and if a MHD model is appropriate, or indeed no symmetry at all. Similarly for resistive wall modes (RWM), axisymmetry is broken by the presence of apertures in the vacuum vessel and reflectional symmetry by the presence of a poloidal divertor. Assuming in either case that the mode is close to ideal, the relevant EBiT model is TaB with either $Z(2)$ symmetry or no symmetry. These models are respectively [48, § 7.3]:

$$\begin{aligned} \dot{a} &= b \\ \dot{b} &= \gamma_1 a + \gamma_2 b + \mu_1 a^3 + \mu_2 a^2 b \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \dot{a} &= b \\ \dot{b} &= \gamma_1 + \gamma_2 b + \mu_1 a^2 + \mu_2 ab \end{aligned} \tag{3.20}$$

for unfolding parameters $\gamma_1, \gamma_2$. Simplest normal form models, ie. with minimum number of free parameters, of higher order may be used for topological matching purposes and are given for Equation (3.20) in ref [49]. In the pure ideal case, the model Eqs (3.15)(3.16) with zero Casimir $C$ applies, and corresponds to the vanishing of the coefficients $\gamma_2$ and $\mu_2$ in Equation (3.19). In the ideal case, the breaking of the rotational and reflectional symmetries may also be modelled by introducing separate terms in the real and imaginary parts of $a$ in the Lagrangian.

Models without symmetry are expected to be relevant to secondary bifurcations, ie. where a configuration with a non-axisymmetric plasma component exhibits further instability. These will most likely be described by the trans-critical or (Andronov-)Hopf bifurcations for which the equations are available in many elementary text-books, viz. respectively

$$\dot{a} = \gamma a + \mu_1 a^2 \tag{3.21}$$

and

$$\begin{aligned} \dot{r} &= r(\gamma + \mu_1 r^2) \\ \dot{\xi} &= \mu_2 \end{aligned} \tag{3.22}$$

where all quantities are real. By substituting $a \to b + b_0$, Equation (3.21) may be transformed to Equation (3.17) with the term in $r^2$ omitted. The models Eqs (3.21)(3.22) should be qualitatively (topologically) correct for the similarity solutions mentioned in Section 3.3.1.

### 3.3.6 More complex mode interactions

This section is a general discussion concerning the production of models where more than one mode may be present. The typical cases are compound sawteeth and interactions between two or more tearing modes. As already mentioned in the above examples, the likely EBiT model depends heavily on whether or not reflectional symmetry also applies, ie. whether the equations should be invariant under the symmetry group

$O(2)$ or $SO(2)$. The reflectional symmetry case is easier, because the $O(2)$-invariance forces many terms in the polynomials to vanish, and will be considered first.

The relevant model for two interacting MHD modes is the unfolding of the Takens-Bogdanov bifurcation with $O(2)$ symmetry, studied by Dangelmayr and Knobloch [50]. The truncated model system in complex variables is

$$\dot{a} = b, \quad \dot{b} = \gamma_1 a + \gamma_2 b + \left[\mu_1 |a|^2 + \mu_2 |b|^2 + \mu_3 (a\bar{b} + \bar{a}b)\right] a + \mu_4 |a|^2 b \tag{3.23}$$

where the $\gamma_i$ and $\mu_i$ are real parameters. Note that the above form Equation (3.23) assumes the marginal modes have the same mode number $K_1 = K_2$. This may be appropriate for a model of compound sawteeth, or any situation where both an ideal and a resistive mode with the same wave-number interact. If both modes have vanishing resistivity, a two-mode version of the Lagrangian Eqs (3.15)(3.16) may be a more efficient model:

$$2L(a, \dot{a}, b, \dot{b}, t) = |\dot{a}|^2 + \mu_1 |\dot{b}|^2 + f_3(|a|^2, |b|^2, \bar{a}b + a\bar{b}) \tag{3.24}$$

where the form of the third argument of $f_3$ implies the modes have the same wavenumber. The theory of normal forms for Lagrangian systems is classical, see for example ref [51, § II.4] for the general result for a system with two degrees of freedom.

For application to tearing modes, resistivity is important and it is likely that $K_1 \neq K_2$. The case of bifurcations where $K_1 = 2K_2$ has been studied by Armbruster *et al* [52], who also describe the general functional forms which are allowed and how to construct equivariant equations for arbitrary $K_1/K_2$, see also ref [27, § 12] which exhibits the construction of models for a multiplicity of interacting modes with different wavenumbers.

For non-MHD modes, bifurcations are expected to be of Hopf type and be only $SO(2)$ equivariant. Rigorous topological argument shows that rotating waves will be the outcome, as shown by Rand [53], who goes on to show there is secondary bifurcation to modulated waves, and interestingly, entrainment is not expected. A suitable model is Equation (3.23) with complex parameters.

It is unclear how high an order of bifurcation needs to be considered in general for the tokamak. The tokamak, as pointed out in ref [6] is a four parameter system in its simplest form, but two of the parameters, namely plasma minor and major radius are highly constrained for a given device. The number of freely variable external parameters is important for dynamical systems modelling as it limits the maximum co-dimension of singularity which might be found (although this maximum is not guaranteed to be realised). In practice, the applied field and current are not single parameters, but have spatial distributions which can be varied by changing the current in say the poloidal field coils, and which are characterised by additional parameters such as discharge elongation and triangularity. Neutral beam and electromagnetic wave input, gas puffing, and pumped divertors enable at least additional variations in the shape of the equilibrium, and might further be treated as giving the devices at least one extra parameter, namely the non-thermal (fast-ion) component of current. This implies that bifurcation models of co-dimension greater than or equal to three may be relevant to devices with auxilliary heating.

Unfortunately, although the topological (qualitative) properties of co-dimension two bifurcations without symmetry are well understood and documented in textbooks such as ref [2] (and indeed many of their analytic properties follow from ref [23, § XIV]), the literature on the higher order bifurcations has not been similarly collated. There are three ways in which a third degree of freedom may be employed, namely to eliminate a higher order term in (1) a first order ODE model, (2) a second order model [54], or (3) to lead to a third, simultaneously unstable mode (triple-zero degeneracy). However, the presence of symmetry also implies that higher order terms vanish, meaning that the cases (1) and (2) are already largely covered by the models presented.

The triple-zero degeneracy (3) has been treated by several different authors. All the co-dimension two bifurcations occur as special cases, and the principal interesting result seems to be that Shil'nikov homoclinic orbits occur generically [55, 56], ie. chaotic behaviour is expected. The relevant ODEs for triple-zero have been derived using computer algebra and the simplest representations are given by Yu and Yuan [57], as

$$\begin{aligned} \dot{a} &= b \\ \dot{b} &= c \\ \dot{c} &= \gamma_1 a + \gamma_2 b + \gamma_3 c + \mu_{20} a^2 + \mu_{11} ab + \mu_{02} c^2 + \mu_{200} ac \end{aligned} \tag{3.25}$$

or

$$\begin{aligned} \dot{a} &= b \\ \dot{b} &= \mu_{20} a^2 + \mu_{11} ac + \mu_{02} c^2 + b(\nu_{00} + \nu_{01} c) \\ \dot{c} &= \gamma_1 a + \gamma_2 b + \gamma_3 c + \varpi_{20} a^2 + \varpi_{11} ac + \varpi_{02} c^2 \end{aligned} \tag{3.26}$$

The above Equation (3.25) is of course relevant to the behaviour of the EBiT ELM model Section 3.3.3.

Thanks to symmetry, the two-mode model of Equation (3.24) is of codimension four, hence resistivity and symmetry-breaking effects may be represented by ODEs describing the general, quadruple-zero, codimension four bifurcation. These models may be obtained using computer algebra software as described in ref [58].

# Chapter 4

# Application to Time Series Modelling

This section is intended only to illustrate how the nonlinear modelling of tokamak time series data might proceed in practice, and is not a comprehensive review of the various techniques that have been developed over the past 20 years. It is worth noting that the 1987 report [59] has stood the test of time quite well, but now is in need of updating taking into account subsequent work. Much of this work as described by Smirnov and Bezruchko [60], Gilmore and collaborators [27, 61], Mandic *et al* [62, 63], Abonyi [64], is however concerned with the choice of embedding dimension (quantity $n$ below), which is less critical if the order of the model is known as a priori information from Section 3

## 4.1 Topological Methods of Time Series Modelling

An introduction to the concepts underlying this approach is provided by ref [59]. A very recent, up-to-date example of the application of the topological method (in the spirit of the Drexel school) is the analysis of the solar sunspot cycle by Letellier *et al* [65].

Ref [59] explains how a time series can be modelled as a geometrical object in a two-, three- or higher dimensional Euclidean space by using delay coordinates. Suppose the time series measurements of the observable $a(t)$ are given by

$$X_k = X(t_k), \quad k = 1, \ldots, N \tag{4.1}$$

where $N$ is the total sample size and the $t_k$ are evenly spaced times, $\Delta t$ apart, then $n$-dimensional vectors may be constructed by

$$\mathbf{x}_k = \left(X(t_k), X(t_k + \Delta), \ldots, X(t_k + [n-1]\Delta)\right), \tag{4.2}$$

where $\Delta$ is an integer multiple of $\Delta t$. The idea is that the nonlinear dynamics can be characterised by quantities which are topological invariants of a curve fitted to the $\mathbf{x}_k$.

There are many other ways of constructing such embeddings (ref [61, § 6] lists over half-a-dozen), but delay coordinates have advantages, eg. when the data is noisy or a delay-differential model might be appropriate.

The importance of topological invariants is that they are a generic property of the system, derivable from almost every observable. Moreover, they are unaffected by simple, non-recursive filters [66], including the filter based on the singular value decomposition (SVD) described in ref [59]. Unfortunately in practice, the usefulness of these invariants may be severely limited by poor choice of delay time $\Delta$ and embedding dimension $n$, especially when noise is present. Nonetheless they can yield information, notably the smallest acceptable $n$, which is vital for the construction of more quantitative models described in Section 4.2.

### 4.1.1 The Drexel Approach

It is of interest to summarise the steps taken by Letellier *et al* [65] to establish a low dimensionality for a time series. They start by noting that it has proved difficult to fit equations to the the sunspot time series, a situation not unlike that in fusion plasmas, cf. refs [1, § 6.3] and [36]. One especial virtue of their method is that it can take a rectified time series measure of magnetic field such as sunspot number and change its sign appropriately near absolute minima. Another interesting point is that the sunspot number is subject to a nonlinear, square-root, transformation before analysis commences.

Quite demanding is the suggestion in ref [65] that data should be sampled at the rate of order $100$ samples per cycle, whereas the earlier work [59] suggested approximately $10$ points to be adequate, a result which may still in fact hold true in many cases as discussed in ref [61, § 6]. A simple low-pass filter is applied to the samples. There is also a discussion in [65] concerning the issue of non-uniform quality of data, which is unlikely to be relevant to tokamaks, although the segmentation of data depending on whether flat-top has been reached, neutral beams switched on, etc. will be a similar issue. Embedding dimension and time delay are determined on the basis of Cao's 'false nearest neighbour' method [67]. A phase portrait is then drawn for the selected $n$ and $\Delta$ and visually inspected. Surrogate random data is generated and used to confirm that the signal has a nonlinear deterministic component. The procedure is not entirely satisfactory because of the subjective element.

Most of the techniques described in ref [59] assume the presence of a strange attractor. The importance of ref [65] is that this assumption is first checked. Since most of the models presented in Section 3 are not in fact expected to exhibit chaos, the main interest of Letellier *et al*'s work is their qualitative comparison between simple ODE models and the reconstructed phase portrait. Regarding the embedded time series as corresponding to stream- or streak-lines of a vector flow-field, topologically invariant features such as approximate orbits (ie. closed curves) and singular points (ie. stagnation points) are of most interest. If there is evidence for chaos, it probably arises as in the Shil'nikov model, see Section 3.3.6. Topological invariants should be calculated much as in ref [59], and if $n = 3$ then the strange attractor should be characterised topologically using branched manifolds as suggested by ref [27].

### 4.1.2 Alternative Approach

Another possible approach to the topological classification of data might again use delay co-ordinates to represent the data as vectors in a multi-dimensional space. The method which, like the branched manifold approach is also applicable only when $n = 2$ or $n = 3$, would draw from results on the theory of the visualisation of 3-D vector fields. These techniques are in some sense more relevant to the present problem in that they tend to focus on the behaviour of the vector fields at critical points where the field vanishes. However, they usually assume that the vector field is known throughout a volume with relatively high sampling rate, which is not usually the case with an arbitrary time series data-set. For a recent review of the subject see ref [68].

## 4.2 Direct Methods of Time Series Modelling

What is primarily needed in the context of this report is directly to find equations which are a global fit to the data. Previous work reviewed in the handbook chapter [60], has considered a considerable range of possibilities for such global equations, although most involve the use of systems of first order ODEs, with time derivatives given by polynomials in the variables just as in Section 3. Some work uses rational polynomials, other work allows for explicit time dependence of the derivatives. This phase of analysis is more challenging than the preceding, since (1) the dependence of the diagnostic on mode amplitude may itself be nonlinear, and (2) the polynomial form of the differential equations may not be an efficient representation in delay coordinates. However, there are examples of its successful application to observational data [69].

Smirnov and Bezruchko [60] distinguish three levels of knowledge about the system under study, namely “black box”, “grey box” and “white box”. A “white box system is one where the relevant terms in the model are known, only the coefficients are to be determined, and a “black box” system is one where nothing is known in advance, with the “grey box” systems representing an intermediate state of *a priori* knowledge. It is important to note that “polynomial” models seldom succeed for chaotic or quasi-stationay black box systems, since the polynomial representation generically produces divergent solutions, and despite its sub-title, Tong’s book [4] prefers models which are nonlinear because they involve a threshold function, *not* a polynomial. Rather than use ordinary polynomials, the recommendation of this report is that when quasi-stationay rather than transient behaviour is expected, to use rational polynomials instead, where the denominator has at least as high an order as the numerator, as this should be very helpful for producing non-divergent models.

Results from initial topological analysis as described in Section 4.1 should have served to narrow down the possible equations and give a good idea of what the leading order terms in the polynomial expansions of the time derivatives should be. Hence in the above jargon, it is expected that a “grey box” analysis will be appropriate. Using the SVD filter will also have given an estimate of the noise level in the data, which will guide selection of the relevant embedding approach. For example if the noise level is very low, an embedding using (approximations of) $\{a, \dot{a}, \ddot{a}, \ldots\}$ is likely to be best.

Given the noise level expected from ref [59], the recommended approach is first to filter, using the SVD approach or a local averaging technique. Thereafter, to fit the filtered data, starting with the simplest rational polynomial model and gradually adding terms until a cost function such as the Akaike criterion (see eg. [4, § 5.4.2]) is minimised. There is overlap with the topological procedures in Section 4.1 since the cost of a model will normally depend on the embedding dimension. The coefficients of the polynomials are usually fitted using least squares algorithms [60]. The likely error in the resulting model is quantified and used as one of the inputs to the Akaike criterion. The model should be validated by, for example, using it to 'predict' data not used in the model derivation.

The global rational polynomial models obtained can be transformed to canonical normal form using computer algebra software, typically either written in Maple. Alternatively a numerical reduction may be performed using pre-existing Matlab$^{TM}$ toolboxes. Of the latter, the publicly available MATCONT software seems to be the most suitable. These canonical forms can be further reduced to simplest normal form (eg. [57]).

If the "white/grey box" approach fails, the "black box" approach replaces polynomials with functions such as radial basis functions and CVA (canonical variate analysis) [70] following Larimore. Alternatively, neural network prediction may be possible, although CVA might be preferred as yielding parameters which might subsequently be used to compare with deterministic models developed at a later date. In either event, the "black box" models produced may have sufficient predictive power to be useful for control purposes.

# Chapter 5

# Summary

This report has laid out a set of linear and nonlinear models for tokamak behaviour close to the onset of large-scale instability, by which is meant instabilities which have low toroidal and poloidal mode numbers rather instabilities which necessarily terminate discharges. It has also indicated how they may be compared with experiment, concentrating on the harder, nonlinear case. Further work will obviously be to compare these theories with experiment. Encouragement to pursue the comparison with nonlinear models in particular is given by the agreement of a linear theory with experiment [36], together with the results of the older work on chaotic models [71, 72, 73]. It is also highly significant that predator-prey systems have been proposed as models for two of the phenomena (fishbones and ELMs), because the finite equilibrium solution of the standard predator-prey or Lotka-Volterra system [74, § 5.2] is an example of an unfolding of TaB [2, § 8.7]. (The van der Pol equation is also similarly an example of TaB, and van der Pol was mentioned in the context of an ELM model [42].)

A key reason for pursuing this work is that, even if all the models should fail to fit the data, this is in itself an interesting result, as it would indicate that tokamak plasmas evolve very differently from experiments with fluids that mimic important aspects of their behaviour. Given the fact that modern experimental plasma data provides strong evidence for the importance of a small number of low-number modes in many cases, it would in fact be quite surprising. Probably, the most likely outcome is an indication of those situations where these lower order models fail to represent the dynamics adequately and hence where there is a need to apply models with less restrictive assumptions such as lattice gases and sandpiles [75], or based on the Ginzburg-Landau theory of modulational instabilities [10, § 7]. Indeed, recent data from the MAST and JET tokamaks indicate an inverse cascade of fluctuations, where it may be appropriate to have a model which switches at a certain time in its evolution from micro-turbulence to the kind discussed here.

If as expected, the modelling is successful in many cases, it can, apart from confirming whether low order dynamics are the key ingredient, also indicate whether the dynamics are:

1. linear or nonlinear

2. ideal or non-ideal

3. constrained by symmetry and what the group is

If the system is linear, it may be possible to discover whether the appropriate theory is normal (meaning self-adjoint) or not. If the system is nonlinear, it may be possible further to extract the nonlinear coefficients which generalise the linear equivalent, growth rate, and permit quantitative agreement with computations based on the solution of partial differential equations. All this information is potentially useful, not only to compare with physical theories, but also in the devising of feedback control strategies to suppress mode growth.

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